

# Approximate rogue wave solutions of the forced and damped Nonlinear Schrödinger equation for water waves

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## Abstract

We consider the effect of the wind and the dissipation on the nonlinear stages of the modulational instability. By applying a suitable transformation, we map the forced/damped Nonlinear Schrödinger (NLS) equation into the standard NLS with constant coefficients. The transformation is valid as long as  $|\Gamma t| \ll 1$ , with  $\Gamma$  the growth/damping rate of the waves due to the wind/dissipation. Approximate rogue wave solutions of the equation are presented and discussed. The results shed some lights on the effects of wind and dissipation on the formation of rogue waves.

*Keywords:* rogue waves, water waves, breathers

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## 1. Introduction

Modulational instability, also known as the Benjamin-Feir instability in the water wave community, has been discovered in the late sixties independently by Benjamin and Feir [1] and Zakharov [2] (see [3] for an historical review on the subject and possible applications). It describes the exponential growth of an initially sinusoidal long wave perturbation of a plane wave solution of the one dimensional water wave problem. For water waves the condition of instability in infinite water depth is that  $2\sqrt{2}a_0k_0N > 1$ , where  $a_0$  is the amplitude of the plane wave and  $k_0$  is the corresponding wave number;  $N = k_0/\Delta K$  is the number of waves under the perturbation of wavenumber  $\Delta K$ . The modulational instability is frequently studied within the Nonlinear Schrödinger (NLS) equation that describes weakly nonlinear and dispersive waves in the narrow band approximation. In this context,

the nonlinear stages of the modulational instability are described by exact solutions of the NLS, known as Akhmediev breathers [4, 5]. Other exact NLS solutions which describe the focussing of an initially non-small perturbation have been derived in [6, 7]. Such solutions have been considered as prototypes of rogue waves [8, 9].

Within the one dimensional NLS equation, the modulational instability is well understood. What is probably less clear is the modulation of waves and the formation of rogue waves in forced (by wind) or damped (by dissipation) conditions. In this regard in the past there has been a number of experimental works, [10, 11, 12], which did not gave a clear picture on the effect of the wind on the modulational instability. A careful discussion of the discrepancy of the results presented in the above papers can be found in [12]. According to their discussion the role of the wind is twofold: i) the wind changes the growth rate of the instability; ii) the natural selection of the sideband frequency is altered with respect to the no wind conditions.

Concerning damping effects, it has been showed in [13] that any amount of dissipation stabilizes the modulational instability, questioning the role of the modulational instability in the formation of rogue waves, [14]. More recently, the role of dissipation and wind in the modulational instability has been considered together within the NLS equation, [15] (then confirmed by fully nonlinear simulations, [16]). The authors performed a linear stability analysis and numerical simulations and found that, in the presence of wind, young waves are more sensitive to modulational instability than old waves.

The just mentioned numerical results (except the one in [16]) are all based on the following forced and damped Nonlinear Schrödinger equation:

$$i\frac{\partial A}{\partial t} - \alpha\frac{\partial^2 A}{\partial x^2} - \beta|A|^2A = i\Gamma A. \quad (1)$$

$A$  is the wave envelope,  $\alpha$  and  $\beta$  are two coefficients that depend on the wavenumber,  $k_0$ , of the carrier wave. The right-hand side is responsible for the forcing,  $\Gamma > 0$ , and/or dissipation,  $\Gamma < 0$ . The two effects are additive so that  $\Gamma$  is in general the sum of forcing coefficient plus a damping one. The wind forcing depends on the ratio between air and water density and the dissipation on the water viscosity, therefore the absolute value of  $\Gamma$  is always a small quantity. Finding analytical solutions of equation (1) is not an obvious task. In the present paper we take advantage of the smallness of  $\Gamma$  and, after a suitable transformation, we are able to find breather solutions of

the forced-damped NLS equation. In the following sections we first describe the transformation and then present the rogue wave analytical solutions.

## 2. Reduction of the forced/damped NLS to the standard NLS

We considered the NLS equation discussed in [15]

$$i\left(\frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial x}\right) - \frac{1}{8} \frac{\omega_0}{k_0^2} \frac{\partial^2 A}{\partial x^2} - \frac{1}{2} \omega_0 k_0^2 |A|^2 A = i\Gamma A \quad (2)$$

with

$$\Gamma = \frac{1}{2g\kappa^2} \frac{\rho_a}{\rho_w} \gamma \omega_0 \left(\frac{u_*}{c}\right)^2 - 2\nu k_0^2 \quad (3)$$

here  $\kappa$  is the Von Karman constant and  $u_*$  is the friction velocity,  $g$  is the gravity acceleration,  $\rho_a$  and  $\rho_w$  are the air and water density, respectively;  $\gamma$  is a coefficient to be determined from the solution of Rayleigh equation associated to the stability of the wind wave problem (see also [17] for a justification of the wind forcing term);  $c$  is the phase velocity,  $\nu$  is the water kinematic viscosity. In [15] the equation is written in a nondimensional form and the coefficient  $K = \Gamma/\omega_0$  is introduced). The surface elevation is related to the envelope as follows:

$$\eta(x, t) = \frac{1}{2} \left( A(x, t) \exp[i(k_0 x - \omega_0 t)] + c.c \right). \quad (4)$$

Note that we use a different definition of the surface elevation from the one in [15] where the 1/2 factor is not included (the consequence is that the coefficient in the nonlinear term in equation (2) differs by a factor of 4 from the one in equation (3.1) in [15]). If  $\epsilon$  is the small parameter in the derivation of the NLS, then it is assumed that the right-hand side term in (2) is of the order of  $\epsilon^2$  as the nonlinear and the dispersive term.

We consider the following new variable:

$$B(x, t) = A(x, t) e^{-\Gamma t} \quad (5)$$

and by selecting a coordinate system moving with the group velocity we get:

$$i \frac{\partial B}{\partial t} - \alpha \frac{\partial^2 B}{\partial x^2} - \beta \exp^{2\Gamma t} |B|^2 B = 0 \quad (6)$$

were  $\alpha$  and  $\beta$  are the coefficients of the dispersive and nonlinear term, respectively. Written in the above form the effect of the forcing/damping term enters as a factor in front of nonlinear term and has the role of enhancing/decreasing the nonlinearity of the system as the wave evolve in time. Recalling that  $\Gamma$  is usually small, we Taylor expand the exponential and re-write the equation as follows:

$$i\frac{\partial B}{\partial t} - \alpha\frac{\partial^2 B}{\partial x^2} - \beta p(t)|B|^2 B = 0 \quad (7)$$

with  $p(t) = 1/(1 - 2\Gamma t)$ . Let's introduce the following change of coordinates:

$$\chi(x, t) = p(t)x, \quad \tau(t) = p(t)t \quad (8)$$

and scale the wave envelope function  $B$  as follows

$$\psi(\chi, \tau) = \frac{B(x, t)}{\sqrt{p(t)}} \exp \left[ -i \left( \frac{\Gamma p(t)x^2}{2\alpha} \right) \right]. \quad (9)$$

After the transformation, the equation (6) results in:

$$i\frac{\partial \psi}{\partial \tau} - \alpha\frac{\partial^2 \psi}{\partial \chi^2} - \beta|\psi|^2 \psi = 0 \quad (10)$$

i.e., the NLS equation with constant coefficients. We have transformed the forced/damped NLS equation into the standard NLS equation whose solutions can be studied analytically. From a physical point of view the transformation (and consequently the validity of the solutions) is valid as long as  $2|\Gamma t| \ll 1$  (the transformation is singular for  $2|\Gamma t| = 1$ ). We underline that the transformation of the forced/damped NLS equation to the standard one has been possible only for  $1/p(t)$  equal to a linear function in  $t$ . For other functional dependences, the transformation does not seem to be possible. Our result is consistent with ones reported in [18, 19] where analytical solutions of the variable coefficient NLS equation are described.

### 3. Rogue wave solutions

In the following we will present three analytical solutions corresponding to the Peregrine, the Akhmediev and the Kuznetsov-Ma breathers for the standard NLS.

The *Peregrine solution* also known as rational solution, has been originally proposed in [5]. It has the peculiarity of being not periodic in time and in space: it is a wave that “appears out of nowhere and disappears without trace” [20, 21]; its maximum amplitude reaches three times the amplitude of the unperturbed waves. For the above reasons it has been considered as special prototype of freak wave, [21]. The Peregrine solution has been recently reproduced experimentally in wave tank laboratories [22] and in optical fibers [23]. Below we present an exact analytical solution of equation (7) which is the analogous of the Peregrine solutions but for the forced/damped case:

$$B(x, t) = B_0 G(x, t) \left( \frac{4(1 - i2\beta B_0^2 p(t)t)}{\alpha + \alpha(2\beta B_0^2 p(t)t)^2 + 2\beta B_0^2 (p(t)x)^2} - 1 \right) \quad (11)$$

with

$$G(x, t) = \sqrt{p(t)} \exp \left[ i \left( \frac{\Gamma p(t)x^2}{2\alpha} - \beta B_0^2 p(t)t \right) \right]. \quad (12)$$

In figure 1 we show an example of such solution for steepness 0.1 and forcing coefficient  $K = 0.0004$  (the same value has been used in [15]). The axis are normalized by the wave period, the wavelength and the initial wave amplitude  $B_0$ . The effect of the wind/dissipation is to increase/reduced the amplitude of the plane wave. As in the case of the standard NLS, the wave appears only once in time and space.

The *Akhmediev solution* [4] describes the modulational instability in its nonlinear regime; it is periodic in space. It is characterized by an amplification factor which ranges from 1 to 3 (this last value corresponds to the Peregrine solution). In the presence of a forcing or damping, the breather has the following analytical form:

$$B(x, t) = B_0 G(x, t) \left( \frac{\sqrt{2}\tilde{\nu}^2 \cosh[\sigma p(t)t] - i\sqrt{2}\tilde{\sigma} \sinh[\sigma p(t)t]}{\sqrt{2} \cosh[\sigma p(t)t] - \sqrt{2 - \tilde{\nu}^2} \cos[\nu p(t)x]} - 1 \right) \quad (13)$$

and

$$\nu = \frac{k_0}{N}, \quad \tilde{\nu} = \frac{\nu}{B_0} \sqrt{\frac{\alpha}{\beta}}, \quad \tilde{\sigma} = \tilde{\nu} \sqrt{2 - \tilde{\nu}^2}, \quad \sigma = \beta B_0^2 \tilde{\sigma}. \quad (14)$$

The function  $G(x, t)$  is reported (12). It should be noted that the function is periodic in space with a period that changes in time. In figure 2 we show an example of such solution for steepness 0.1,  $N = 5$  and forcing coefficient of  $K = 0.0004$ .

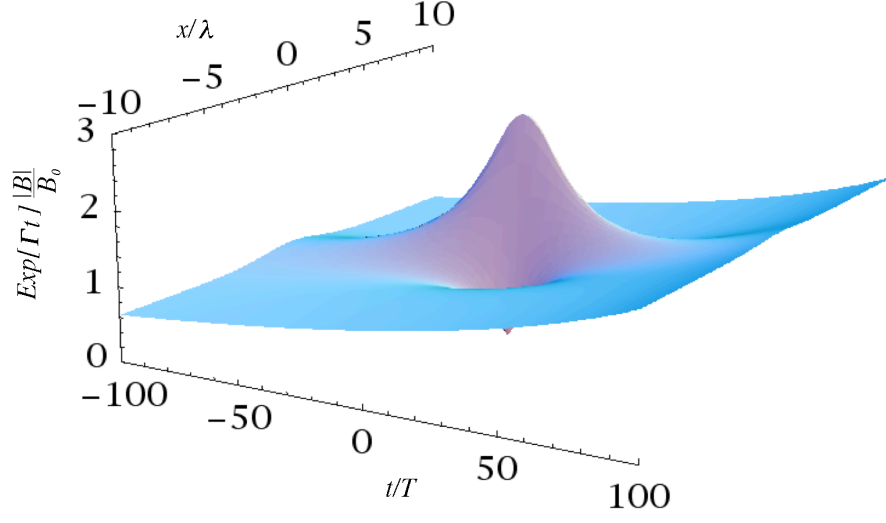


Figure 1: The Peregrine solution of the forced NLS equation.

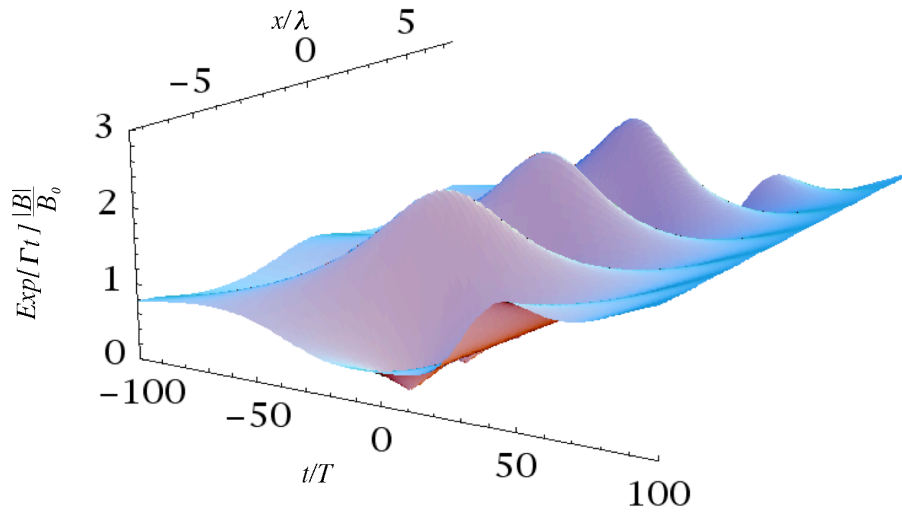


Figure 2: The Akhmediev solution of the forced NLS equation.

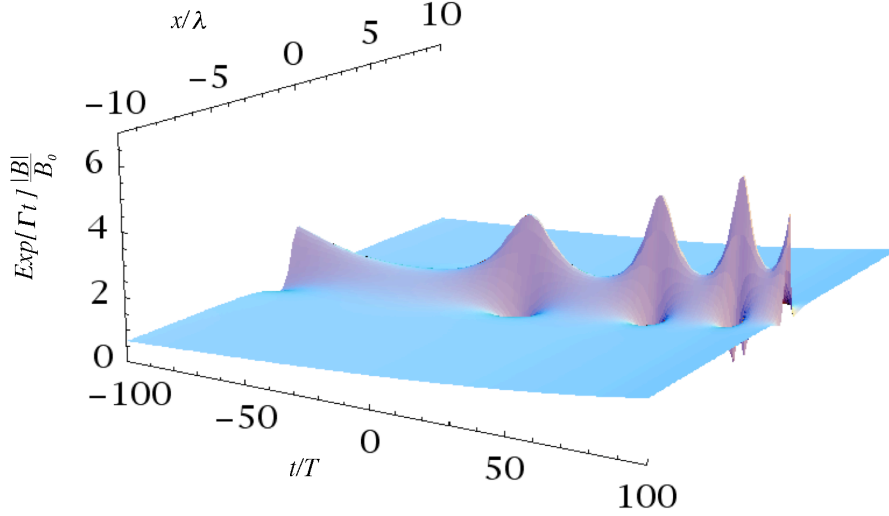


Figure 3: The Kuznetsov-Ma solution of the forced NLS equation.

The *Kuznetsov-Ma solution* [6] is periodic in time and decrease exponentially in space. While for the Akhmediev breather the large time (positive or negative) limit is a plane wave plus a small perturbation, the modulation for the Ma breather is never small. The solution for the forced/damped equation is here reported:

$$B(x, t) = B_0 G(x, t) \left( \frac{-\sqrt{2}\tilde{\mu}^2 \cos[\rho p(t)t] + i\sqrt{2}\tilde{\rho} \sin[\rho p(t)t]}{\sqrt{2} \cos[\rho p(t)t] - \sqrt{2 + \tilde{\mu}^2} \cosh[\nu p(t)x]} - 1 \right) \quad (15)$$

with

$$\mu = B_0 \tilde{\mu} \sqrt{\frac{\beta}{\alpha}}, \quad \tilde{\rho} = \tilde{\mu} \sqrt{2 + \tilde{\mu}^2}, \quad \rho = \beta B_0^2 \tilde{\rho}. \quad (16)$$

$\tilde{\mu}$  is a parameter related to the amplification factor. In figure 3 we show an example of such solution for steepness 0.1,  $\tilde{\mu} = \sqrt{2}$  and forcing coefficient of  $K = 0.0004$ . The periodicity (appearance of maxima) changes in time and increase in the presence of forcing and decrease for the damping case.

#### 4. Conclusion

In the present Letter we have considered the problem of generation of rogue waves in the presence of wind forcing or dissipation. Our work is based on the one dimensional forced/damped NLS equation.

Under the assumption of  $2|\Gamma t| \ll 1$ , where  $\Gamma$  is the forcing ( $\Gamma > 0$ ) or damping ( $\Gamma < 0$ ) term, we have shown how the equation can be mapped in the standard NLS equation with constant coefficients. In this framework, we have found explicit analytical breather solutions.

As mentioned the effect of wind/dissipation is to increase/reduce in time the coefficient in front of the nonlinear term. This has an impact on the modulational instability; in particular, an initially stable (unstable) wave packet could be destabilized (stabilized) by the wind (dissipation). Similar results have been obtained for the interaction of waves and current (see [24]). The present results should be tested in wind waves tank facilities.

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